

Introduction to Hyperbolic Groups

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1 Quasi-isometries

Let $(X, d_X), (Y, d_Y)$ be metric spaces.

Definition 1. A map $f : X \rightarrow Y$ is a (λ, ϵ) -quasi-isometric embedding ($\lambda \geq 1, \epsilon \geq 0$) if for all $x, y \in X$,

$$\frac{1}{\lambda}d_X(x, y) - \epsilon \leq d_Y(f(x), f(y)) \leq \lambda d_X(x, y) + \epsilon. \quad (1)$$

(Note that without ϵ , f is bi-Lipschitz). We say f is a (λ, ϵ) -quasi-isometry if additionally

$$Y = N_\epsilon(f(X)) = \{y \in Y \mid \exists x \in X : d_Y(y, f(x)) \leq \epsilon\}. \quad (2)$$

Property (1) is called coarsely-injective because $\text{diam}_X(f^{-1}(y)) \leq \epsilon \forall y \in Y$, and property (2) is called coarsely-surjective.

Exercise 1. QI is an equivalence relation on metric spaces.

We need to show that if $f : X \rightarrow Y$ is a (λ, ϵ) -QI, then there exists a $g : Y \rightarrow X$ which is a (λ', ϵ') -QI such that $f \circ g$ and $g \circ f$ are bounded from identity maps. We call g the quasi-inverse of f , and we say $X \xrightarrow{\text{Q.I.}} Y$ if there exists an $f : X \xrightarrow{\text{Q.I.}} Y$.

Exercise 2. $QI(X)/\sim$ is a group. Here, $QI(X) = \{f : X \xrightarrow{\text{Q.I.}} X\}$, and $f \sim g$ if f and g are bounded from each other, i.e. $\sup_{x \in X} d(f(x), g(x)) < \infty$.

Some important examples:

- $X \cong \{*\} \Leftrightarrow \text{diam}(X) < \infty$
- $\mathbb{Z} \xrightarrow{\text{Q.I.}} \mathbb{R}$
- $[0, \infty) \rightarrow \mathbb{R}^2 : t \mapsto te^{i2\pi \log(1+t)}$ [Logarithmic spiral]

Exercise 3. Show the Logarithmic spiral is a QI-embedding, i.e. a quasi-geodesic ray.

Definition 2. A (λ, ϵ) -quasi-geodesic (segment/ray/biinfinite) in X is a (λ, ϵ) -QI embedding $c : I \rightarrow X$ where $I \subseteq \mathbb{R}$.

Quasi-geodesics have the following properties:

- They need not be continuous because of additive error, and we can define $c : I \cap \mathbb{Z} \rightarrow X$.
- They are almost continuous.

Lemma 1. *If $c : I \rightarrow X$ is a (λ, ϵ) -quasi-geodesic and let $x = c(t)$. Then either x has distance at most $\epsilon + 1$ from one of the endpoints of $c(I)$, or there exist $t_1 < t < t_2$ such that $d(c(t_i), x) \leq \epsilon + 1$ for $c(t_i) \neq x$, $i = 1, 2$. (Exercise)*

2 Geodesic Metric Spaces

Definition 3. *A metric space (X, d) is a geodesic metric space if for all $x, y \in X$, there exists an isometric embedding $c : [0, d] \rightarrow X$ such that $c(0) = x$, $c(d) = y$, and $d = d(x, y)$.*

Primary examples:

- Riemannian manifolds
- Metric graphs: identify each edge from $[0, 1]$
- Polyhedral complexes: cells are convex polytopes (in $\mathbb{R}^n, \mathbb{H}^n, \dots$) and gluing maps are isometries along faces. This is a generalization of metric graphs. i.e. cube complexes

Exercise 4. *You can model any geodesic metric space by a metric graph up to QI.*

Definition 4. *Let $G = \langle S \rangle$, G is a group generated by S . We say S is symmetric if $g \in S \iff g^{-1} \in S$. The word metric on G is*

$$\|g\|_S := \min\{k \in \mathbb{N} \cup \{0\} : g \in S^k\}.$$

For a group G generated by a set S , define $\text{Cay}(G, S)$ to be the Cayley graph where G is the vertex set of $\text{Cay}(G, S)$. The word metric is equivalent to the path metric on the Cayley graph. $d(1, g)$ in $\text{Cay}(G, S)$ is a metric graph.

Properties of Cayley graphs:

- $\text{Cay}(G, S)$ is proper $\iff |S| < \infty$
- $G \curvearrowright \text{Cay}(G, S)$ is transitive on vertices and cobounded, $\text{Cay}(G, S) = G \cdot B$ where B is a bounded set like the unit ball. Note that cobounded can be replaced by cocompact if $|S| < \infty$.

Definition 5. Proper (properly-discontinuous): for all bounded sets $K \subseteq X$, $G \curvearrowright X$,

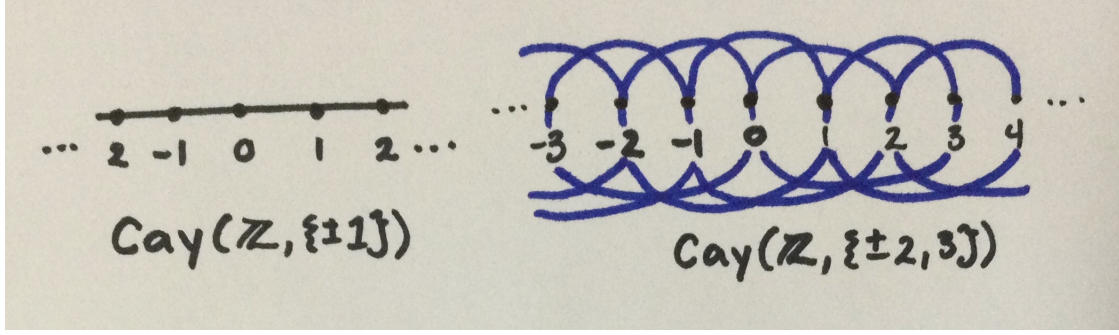
$$\#\{g \in G : g(K) \cap K \neq \emptyset\} < \infty$$

Definition 6. By isometries: $d(x, y) = d(gx, gy)$ for all $g \in G, x, y \in X$

Definition 7. *We say an action $G \curvearrowright (X, d)$ is geometric if G acts by isometries, and the action is cobounded and proper.*

Examples:

- $\text{Cay}(\mathbb{Z}, \{\pm 1\})$
- $\text{Cay}(\mathbb{Z}, \{\pm 2, 3\})$ (in fact, $\mathbb{Z}$ can be generated by any two coprime integers and their inverses)



Note that $\text{Cay}(G, S) \neq \text{Cay}(G, S')$ if $S \neq S'$.

Lemma 2. Let G be finitely generated by S and S' . Then $\text{id} : G \rightarrow G$ extends to a QI $\text{Cay}(G, S) \rightarrow \text{Cay}(G, S)$.

Proof. G = vertex set of the Cayley graphs

$$\text{Cay}(G, S') \stackrel{\text{Q.I.}}{\cong} (G, d_{S'}) \stackrel{\text{exercise}}{\cong} (G, d_S) \stackrel{\text{Q.I.}}{\cong} \text{Cay}(G, S)$$

□

Proposition 1. (Milnor-Schwarz Lemma/The Fundamental Theorem of Geometric Group Theory)

Suppose X is a geodesic metric space, and $G \curvearrowright X$ is geodesic. Then G is finitely generated, and for all $O \in X$, the orbit map $G \rightarrow X : g \mapsto g \cdot O$ is a QI.

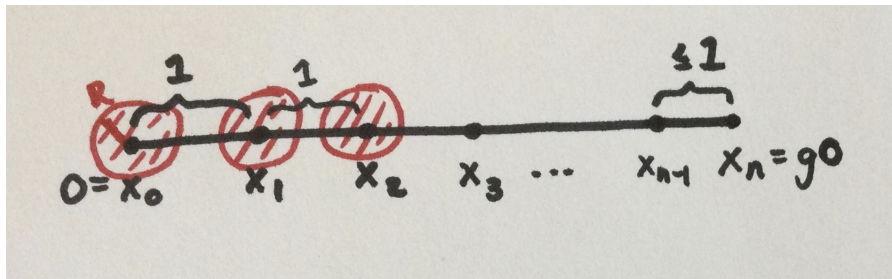
Proof. The action is cobounded implies that $X = G \cdot B_R$ where B_R is a ball centered at $O \in X$ with a radius R . Set $S = \{g \in G : d(O, gO) \leq 2R + 1\}$. Since the action is proper, we know that $|S| < \infty$.

We now have two claims: $G = \langle S \rangle$ and $(G, d_S) \stackrel{\text{QI}}{\cong} X$.

For the upper bound, we let $g \in \langle S \rangle$, so $g = s_1 s_2 \dots s_k$ for $k = \|g\|_S$. Then, by the triangle inequality,

$$\begin{aligned} d(O, gO) &= d(O, s_1 \dots s_k O) \leq d(O, s_1 O) + d(s_1 O, s_2 O) + \dots + d(s_1 \dots s_{k-1} O, s_1 \dots s_k O) \\ &= d(O, s_1 O) + d(O, s_2 O) + \dots + d(O, s_k O) \\ &\leq (2R + 1)k \\ &= (2R + 1)\|g\|_S \end{aligned}$$

For the lower bound and to show that S generates G , we let $g \in G$ and c be a geodesic from O to $g \cdot O$.



Choose $O = x_0, x_1, \dots, x_n = gO$ as above along c where the distance $d(x_i, x_{i+1}) = 1$ for $i \in [0, n - 2]$, and $d(x_{n-1}, x_n) \leq 1$. We see that $n - 1 < d(O, gO) \leq n$. We can choose a radius R that is small relative to 1. Then, for all x_i , there exist $h_i \in G$ such that $d(h_i O, x_i) \leq R$. Set $h_0 = 1$ and $h_n = g$. Then,

$$d(O, h_i^{-1} h_{i+1} O) = d(h_i O, h_{i+1} O) \leq 2R + 1.$$

Therefore, $h_i^{-1} \circ h_{i+1} \in S$. Finally,

$$g = h_n = h_1(h_1^{-1}h_2)(h_2^{-1}h_3)\dots(h_{n-1}^{-1}h_n) \in \langle S \rangle$$

This gives, $\|g\|_S - 1 \leq n - 1$ as the lower bound with additive error 1. Hence, $G \hookrightarrow X$ is a $(2R + 1, 1)$ - QI embedding. Of course, $X = G \cdot B_R \implies \text{Im}(G)$ is coarsely onto. $\square$

Remark. The only place we used $G \curvearrowright X$ is proper is to say $|S| < \infty$. So if we drop this assumption, it is still true that $G = \langle S \rangle$ and $(G, d_S) \stackrel{\text{Q.I.}}{\cong} X$.

Example of a non-proper action: $G = A \rtimes B \curvearrowright T_{\text{Bass-Serre}}$. (This is only proper if both $|A|, |B| < \infty$)

Corollary 1. *If $G = \pi_1(M)$ for M a closed Riemannian manifold, then $G \stackrel{\text{Q.I.}}{\cong} \tilde{M}$*

Corollary 2. *All fundamental groups of closed surfaces with $\chi < 0$ are QI.*

Corollary 3. *Let G be finitely generated and $H \leq G$ be a finite index subgroup. Then H is finitely generated and $H \stackrel{\text{Q.I.}}{\cong} G$.*

Corollary 4. *If G is finitely generated and $N \trianglelefteq G$ is a finite normal subgroup, then G/N is finitely generated and $G/N \stackrel{\text{Q.I.}}{\cong} G$.*

Fact: Let T_n be the regular tree with valence n at every vertex. Then $T_n \stackrel{\text{Q.I.}}{\cong} T_m$ for all $m, n \geq 3$. In fact, all trees of bounded valence to each other are QI.

Corollary 5. *All non-abelian free groups are QI. $\text{Cay}(\mathbb{F}_n, \{\text{basis}\}) = T_{2n}$*

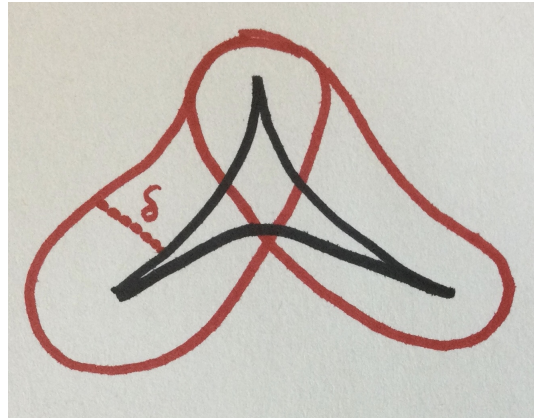
Fact: G is virtually $\mathbb{Z}$ (i.e. contains $\mathbb{Z}$ as a finite index subgroup) $\iff G \curvearrowright \mathbb{R}$ geometrically.

3 Gromov Hyperbolic Spaces

Let (X, d) be a geodesic metric space.

Definition 8. *Let $\alpha_0, \alpha_1, \alpha_2$ be the sides of a geodesic triangle $\Delta \subseteq X$. We say Δ is δ -slim if $\alpha_i \subseteq N_\delta(\alpha_{i-1}) \cup N_\delta(\alpha_{i+1})$.*

We say X is δ -hyperbolic if every Δ is δ -slim for some δ .



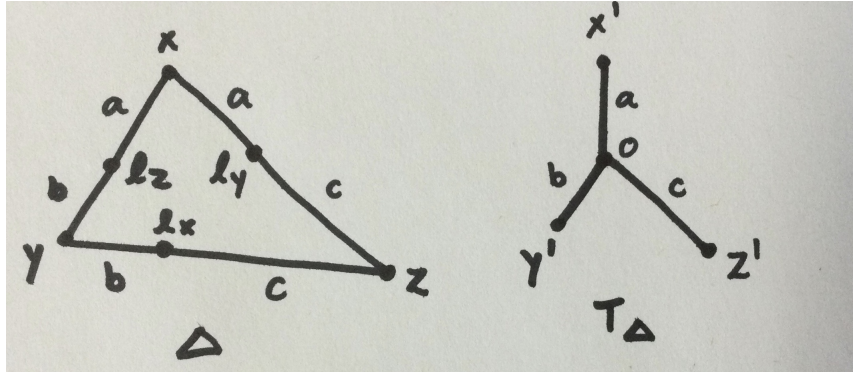
Examples:

- Bounded metric spaces ($\delta = \text{diam}(X)$)
- Trees are 0-hyperbolic
- Hyperbolic spaces $\mathbb{H}^n$ (it is enough to show that $\mathbb{H}^2$ is δ -hyperbolic)

Non-example: $\mathbb{R}^2$

We can think about δ -slim triangles as fat trees using the comparison tripod. Given $x, y, z \in X$, there exist unique $a, b, c \geq 0$ such that $d(x, y) = a + b$, $d(x, z) = a + c$, and $d(y, z) = b + c$.

There exists a unique map $\chi_\Delta : \Delta \rightarrow T_\Delta$ such that $x \mapsto x'$, $y \mapsto y'$, $z \mapsto z'$, and it restricts to an isometry along each side of Δ . Call D_Δ the central vertex of the tripod. Then $\text{Insize}(\Delta) = \text{diam}(\chi_\Delta^{-1}(D_\Delta))$.



Definition 9. Say Δ is δ -thin if for all $p, q \in \chi_\Delta^{-1}(t)$, $d(p, q) \leq \delta$ for all $t \in T$.

Proposition 2. Let (X, d) be a geodesic metric space. Then the following are equivalent:

1. $\exists \delta_1$ such that every $\Delta \subseteq X$ is δ_1 -slim
2. $\exists \delta_2$ such that every $\Delta \subseteq X$ is δ_2 -thin
3. $\exists \delta_3$ such that $\text{Insize}(\Delta) \leq \delta_3 \forall \Delta \subseteq X$

Proof. Exercise. Hint: 2 $\implies$ 1 is obvious. 1 $\implies$ 3 is triangle inequality. 3 $\implies$ 1 is showing every $p, q \in \chi_\Delta^{-1}(t)$ is the preimage of some central vertex of some tripod for some triangle. $\square$

4 Hyperbolicity and Quasi-Isometry

Definition 10. Given a path $c : [a, b] \rightarrow X$, we define the length of c to be

$$\ell(c) = \sup_{a=t_0, t_1, \dots, t_n=b} \left\{ \sum_{i=1}^{n-1} d(c(t_i), c(t_{i+1})) \right\}$$

We say the c is rectifiable if $\ell(c) < \infty$.

Definition 11. Let $A, B \subseteq X$. Then the Hausdorff distance is

$$d_{\text{Haus}} = \inf \{ \epsilon : A \subseteq N_\epsilon(B), B \subseteq N_\epsilon(A) \}$$

Lemma 3. (*Taming quasi-geodesics*) For X a geodesic metric space, let $c : [a, b] \rightarrow X$ be a (λ, ϵ) -quasi-geodesic. Then there exists a $(\lambda, 2(\lambda + \epsilon))$ -quasi-geodesic $c' : [a, b] \rightarrow X$ with the following properties:

1. c' is continuous with $c'(a) = c(a)$ and $c'(b) = c(b)$
2. c' is rectifiable and for all $a \leq t < t' \leq b$,

$$\ell(c'|_{[t, t']}) \leq k_1 d(c'(t), c'(t')) + k_2$$

where k_1, k_2 depends on λ and ϵ

3. $d_{Haus}(Im(c), Im(c')) \leq \lambda + \epsilon$

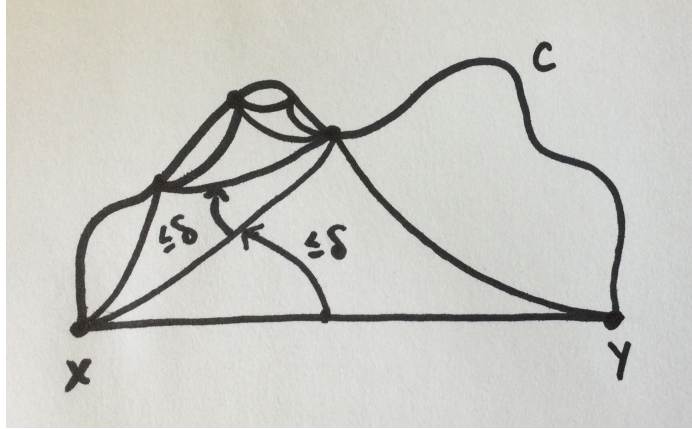
Proof. (Idea) Set c' to agree with c on $[a, b] \cap \mathbb{Z}$. Then connect the dots. □

Lemma 4. Let X be a δ -hyperbolic space, and let c be a continuous rectifiable path from $x \in X$ to $y \in X$. Then, for every geodesic $[x, y]$ and any point $z \in [x, y]$,

$$d(z, Im(c)) \leq \delta |\log_2(\ell(c))| + 1$$

Remark: The hypothesis that X is δ -hyperbolic is a necessary condition. This lemma is false in Euclidean space.

Proof. If $\ell(c) \leq 1$, then we are done. Assume $\ell(c) > 1$. Find the midpoint of c , and form a geodesic triangle with this midpoint and the endpoints of c , x and y . Since this triangle is geodesic, one of the sides must be a length at most δ from the base of the triangle. Without loss of generality, choose a side with this property. Now repeat this process inductively using the chosen side as the base of the triangle as shown below. After $\log_2(\ell(c))$ many steps, we will have the endpoints of our triangle at most length 1, which gives us the desired inequality.



□

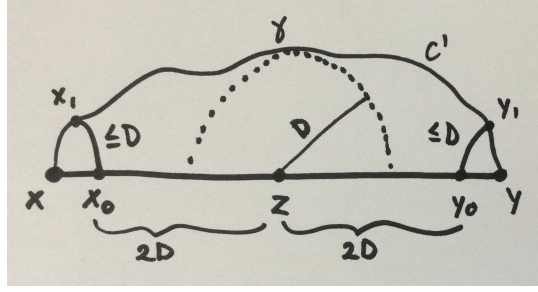
Theorem 1. (*Morse Lemma*) For all $\delta, \lambda, \epsilon$, there exists an $M = M(\delta, \lambda, \epsilon)$ satisfying the following:

Let c be a (λ, ϵ) -quasi-geodesic in X , a δ -hyperbolic space, connecting $x, y \in X$. Then, for every geodesic sequence $[x, y]$, $d_{Haus}([x, y], Im(c)) \leq M$.

Proof. Tame c by c' so that $d_{Haus}(Im(c), Im(c')) \leq \lambda + \epsilon$. Let

$$D = \sup_{z \in [x, y]} \{d(z, Im(c))\}$$

which is realized by some z by continuity.

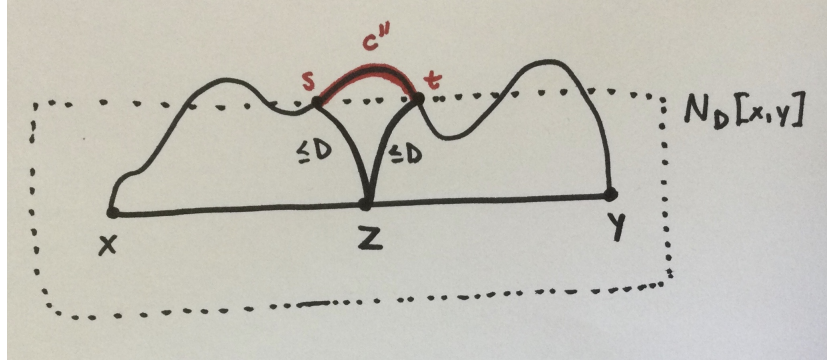


We see here that $d(x, y) \leq 6D$. Then by tameness of c' , we get that $\ell(\gamma) \leq 6Dk_1 + k_2$ for k_i depending on λ and ϵ . Now, consider $\gamma' = [x_0, x_1] \cup \gamma \cup [y_1, y_0]$. We have an open ball $B_D(z)$ with $B_D(z) \cap \text{Im}(\gamma') = \emptyset$. Then,

$$D \leq d(z, \text{Im}(\gamma')) \leq \delta \log_2(\ell(\gamma')) + 1 \leq \delta \log_2(6Dk_1 + 2D + k_2) + 1.$$

Therefore, D is bounded in terms of λ , ϵ , and δ , and $[x, y] \subseteq N_D(\text{Im}(c'))$.

For the other containment, choose $D' = D(1 + k_1) + \frac{k_2}{2}$.



Let c'' be a maximal segment lying outside of $N_D([x, y])$. We know $d(c''(s), c''(t)) \leq 2D$, so by tameness, $\ell(c'') \leq 2Dk_1 + k_2$, and so every point in c'' is at most D' away from $[x, y]$. The triangle inequality then gives us

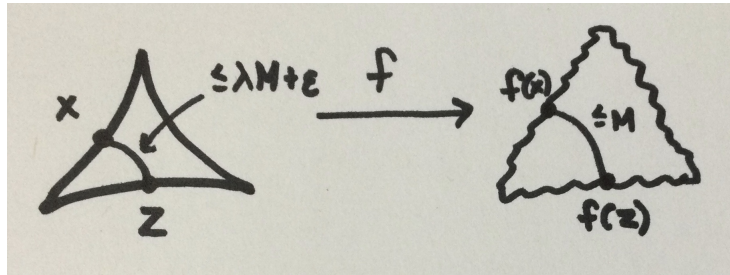
$$d_{\text{Haus}}([x, y], \text{Im}(c)) \leq d_{\text{Haus}}([x, y], \text{Im}(c')) + \lambda + \epsilon = D_1 + \lambda + \epsilon.$$

□

Corollary 6. In a δ -hyperbolic space X , (λ, ϵ) -quasi-geodesic triangles are $M = M(\delta, \lambda, \epsilon)$ slim.

Theorem 2. Gromov Hyperbolicity is a QI invariant. More precisely, suppose $f : X \rightarrow Y$ is a (λ, ϵ) -QI embedding. Then Y is δ -hyperbolic implies that X is δ' -hyperbolic where δ' depends on δ , λ , and ϵ .

Proof. Let $f : X \rightarrow Y$ be a (λ, ϵ) embedding.



□

5 Guessing Geodesic Lemma

Question: How can we show a geodesic metric space X is δ -hyperbolic?

We have a couple of characterizations of this including exponential divergence and (sub-quadratic) linear isoperimetric inequality.

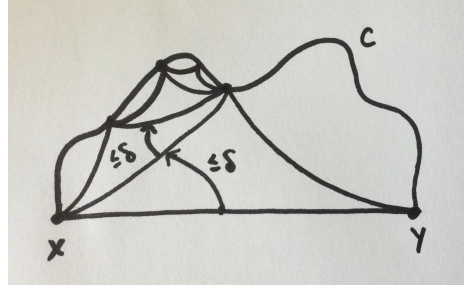
Different versions of Guessing Geodesic Lemma can be found in Masur-Minsky, Bowditch, Hamenstadt, Masur-Schleimer, etc. Here's one version.

Proposition 3. (*Hamenstadt, Bowditch*) Let (X, d) be a geodesic metric space. Suppose there exists an M and for all $x, y \in X$, there exists a path $\eta(x, y)$ connecting them such that

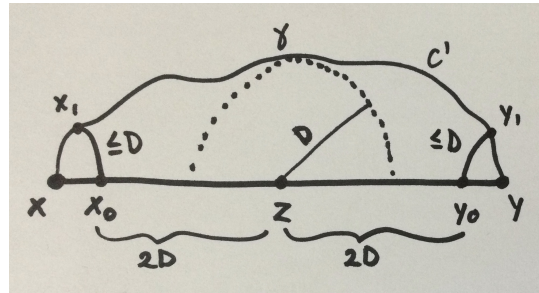
1. If $d(x, y) \leq 1$, then $\text{diam}(\eta(x, y)) \leq M$
2. $\forall x, y, z \in X$, $\eta(x, y) \subseteq N_M(\eta(x, z) \cup \eta(y, z))$, i.e. M -slimness of triangles formed by these paths.

Then X is δ -hyperbolic and $d([x, y], \eta(x, y))$ is uniformly bounded in terms of M for all x, y .

Proof. Step 1 is to show that for every path c connecting x and y , and $z \in \eta(x, y)$, $d_{\text{Haus}}(z, \text{Im}(c)) \leq M|\log_2(\ell(c))| + M$. We show this using the same picture as before, where now the geodesic $[x, y]$ is replaced by $\eta(x, y)$.



Step 2 is the other picture that we previously saw, except now c' is replaced by $[x, y]$, and $[x, y]$ is replaced by $\eta(x, y)$.



□

Remark: The version of the guessing-geodesic-lemma above is closest to the version given by Hamenstadt but without the assumption of "closed under taking subpaths". Bowditch's proof works without that extra assumption and his step 2 of the proof is slightly different from the one given above, but the spirit remains the same.

6 Gromov Hyperbolic Groups

Definition 12. A finitely generated group G is δ -hyperbolic (or *hyperbolic*) if $\text{Cay}(G, S)$ is δ -hyperbolic for some $|S| < \infty$.

Fact: (Gromov) If G is hyperbolic relative to a generating set S , then G is hyperbolic relative to any finite generating set (this is well-defined).

Fact: (Gromov + Milnor-Svarc) G hyperbolic $\iff G$ acts geometrically on some hyperbolic space X .

Primary examples:

- Finite groups
- Virtually- $\mathbb{Z}$
- $\mathbb{F}_n$
- $\pi_1(S)$ for $\chi(S) < 0$
- $\pi_1(M)$ for a closed hyperbolic manifold M or negatively curved manifold
- Small cancellation groups (defined by a presentation $G = \langle S | R \rangle$ satisfying some properties)

Exercise 5. Classify all abelian hyperbolic groups.

Definition 13. A finitely generated subgroup $H \leq G$ for a finitely generated group G is undistorted if $i : H \hookrightarrow G$ is a QI-embedding. i.e. for all $h \in H$, $\|h\|_H \leq \lambda \|h\|_G + \epsilon$. (Note that the bound on the other side is always true.)

Examples:

- Subgroups of abelian groups (exercise)
- Subgroups of free groups

Non-example: Baumslag-Solitar groups

$$BS(m, n) = \langle a, t : ta^mt^{-1} = a^n \rangle$$

In $BS(1, 2) = \langle a, t : tat^{-1} = a^2 \rangle$, $\langle t \rangle$ is an undistorted copy of $\mathbb{Z}$, but $\langle a \rangle$ is (exponentially)-distorted in $BS(1, 2)$.

$$\begin{aligned} a^{2^n} &= t^n a t^{-n} \\ \implies 2^n &= \|a^{2^n}\|_{\mathbb{Z}} \not\leq \|a^{2^n}\|_{\langle a, t \rangle} \leq 2n + 1 \end{aligned}$$

Theorem 3. For all $g \in G^{\text{hyp}}$ of infinite order, $\langle g \rangle$ is undistorted, i.e. $\{g^n\}_{n \in \mathbb{Z}}$ is quasi-geodesic.

Corollary 7. If $g \in G^{\text{hyp}}$ has infinite order and g^m is conjugate to g^n , then $m = n$ and $m = -n$.

Proof. Assume for a contradiction that $|n| > |m|$. Then $g^{n^k} = t^k g^{m^k} t^{-k}$, and

$$\frac{1}{\lambda} |n^k| - \epsilon \leq \|g^{n^k}\|_G \leq 2k \|t\| + \lambda |m|^k + \epsilon$$

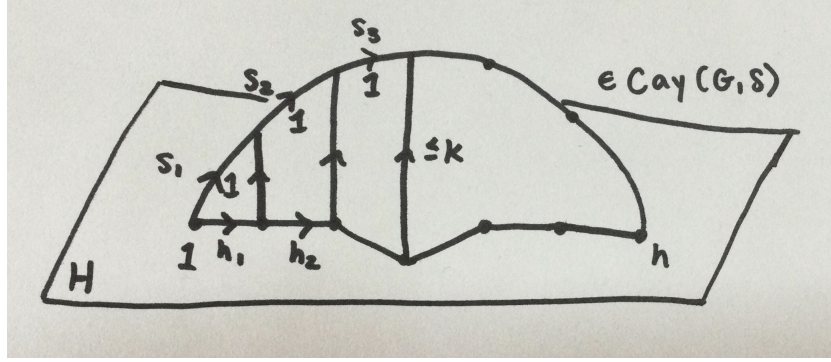
For k large enough, we get a contradiction. □

Corollary 8. There is no copy of $BS(m, n)$ in G^{hyp} .

Proof. If $|m| \neq |n|$, then we are done by above. The case of $|m| = |n|$ is left as an exercise. $\square$

Definition 14. A subset $A \subseteq X$ is quasi-convex if there exists a constant k such that any geodesic $[x, y]$, $x, y \in A$ has $[x, y] \subseteq N_k(A)$.

Lemma 5. Suppose $H \leq G$ is quasi-convex relative to a generating set S . Then H is finitely generated and $H \hookrightarrow G$ is undistorted.



Proof. From the figure above, we see that $\|h_i\|_G \leq 2k + 1$. Let $T = \{h \in G : \|h\| \leq 2k + 1\}$. Then T generates H and $\|h\|_T \leq n = \|h\|_S$. $\square$

Corollary 9. Let G be a hyperbolic group and $H \leq G$ a finitely generated subgroup. Then H is undistorted in $G \iff H$ is quasi-convex relative to a finite generating set of G . In particular, being quasi-convex is well-defined in a hyperbolic group G (independent of a generating set). Quasi-convex subgroups of hyperbolic groups are finitely generated and hyperbolic.

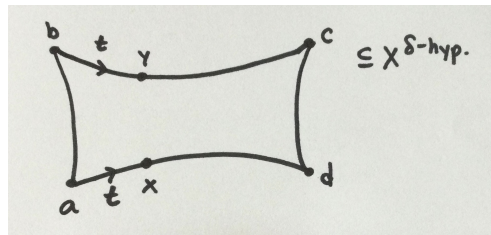
Lemma 6. Suppose $H_1, H_2 \leq G$ are quasi-convex relative to S . Then $H_1 \cap H_2$ is also quasi-convex rel. to S .

Note that the above statement is false more generally for metric spaces.

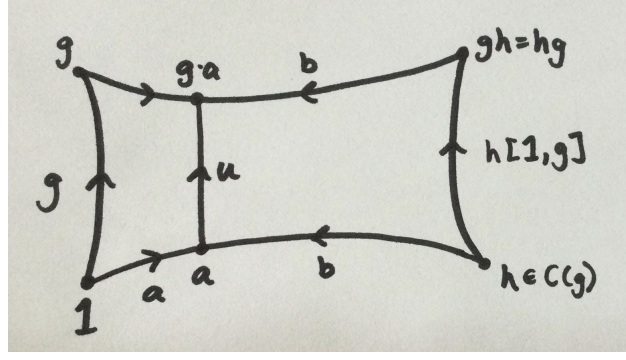
Exercise 6. Show that an intersection of quasi-convex subsets of a metric space X may not be quasi-convex.

Proposition 4. For all $g \in G^{\text{hyp}}$ of infinite order, the centralizer $C(g)$ is quasi-convex.

Exercise 7. Show $d(x, y) \leq \max\{d(a, b), d(c, d)\} + 10\delta$. (We use this exercise in the proof of the proposition).

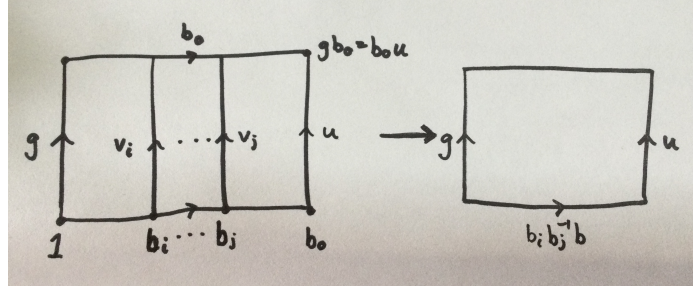


Proof of proposition 4. Now, let $h \in C(g)$. For a fixed $\text{Cay}(G, S)$, consider $Q = (1, h, g, hg = gh)$.



Our goal is to show $\forall a \in [1, h] \cap G, \exists h' \in C(g)$ such that $d(a, h') \leq K$, where u is the path from a to ga . By the exercise above, $\|u\| = d(g, ga) \leq \|g\| + 10\delta$. Now define $\Omega = \{c \in G : cu = gc\}$. Ω is nonempty because $a \in \Omega$, and it's not hard to see that $ac^{-1} \in C(g)$ for all $c \in \Omega$. Let $b_0 \in \Omega$ of minimal length.

We claim $\|b_0\| \leq K = \#B(1, \|g\| + 20\delta)$.



For all $b_i \in [1, b_0] \cap G$, there exists $v_i \in B := B(1, \|g\| + 20\delta)$, such that $b_i v_i = g b_i$. If $\|b_0\| > \#B$, then there exists $i \neq j$ with $v_i = v_j$. Now, "delete" the region between v_i and v_j and glue v_i to v_j , to get a new word $b_i b_j^{-1} b_0 \in \Omega$, but $\|b_i b_j^{-1} b_0\| < \|b_0\|$ which contradicts the minimal length assumption on b_0 . $\square$

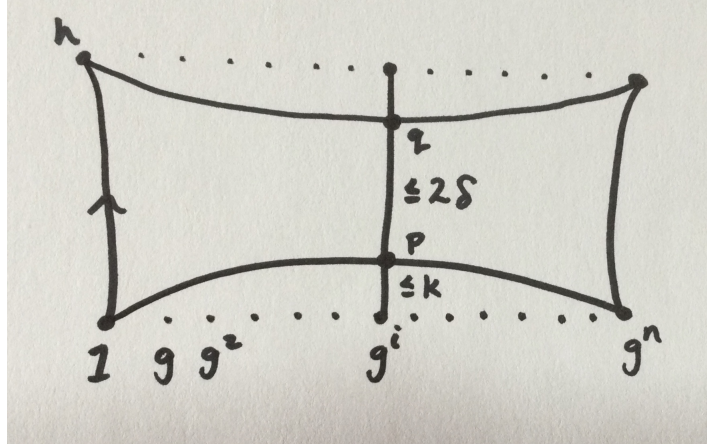
Proof of Theorem 3. Let $g \in G^{hyp}$ be of infinite order. $C(g)$ is quasi-convex, hence it is hyperbolic and finitely generated by some finite set S . This implies that $Z(C(g)) = \bigcap_{h \in S} C(h)$ is quasi-convex, hence hyperbolic. Since $Z(C(g))$ is abelian and hyperbolic, so $Z(C(g))$ is virtually $\mathbb{Z} \simeq \langle g \rangle$. We get the following chain of maps:

$$\langle g \rangle \xrightarrow{\text{QI}} Z(C(g)) \xrightarrow{\text{QI emb.}} G.$$

This gives our desired QI embedding $\langle g \rangle \hookrightarrow G$. $\square$

Corollary 10. Suppose $g \in G^{hyp}$ of infinite order. Then $\langle g \rangle$ has finite index in the centralizer $C(g)$. In particular, $\mathbb{Z} \times \mathbb{Z} \not\leq G$ is not a subgroup.

Proof of corollary 10. We want to show that for all $h \in C(g)$, there exists a g^m such that $\|hg^m\| \leq R$, for some uniform R .



Find an n such that $\|g^n\| > 2\|h\| + 2\delta$. Then, for the midpoint $p \in [1, g^n]$, there exists $q \in h[1, g^n]$ such that $d(p, q) \leq 2\delta$. Then, there are i, j such that $d(p, g_i) \leq k$ and $d(q, gp_j) \leq k$, where k is the quasi-convexity of g . This gives

$$d(1, hg^{-i}g^j) = d(g^i, hg^j) \leq 2k + 2\delta = R.$$

□

Corollary 11. *The center of a hyperbolic group, $Z(G)$, is virtually cyclic.*

Corollary 12. *If $G > \mathbb{F}_2 = \langle a, b \rangle$, then $|Z(G)| < \infty$.*

Fact: If $|G^{hyp}| = \infty$, then there exists a $g \in G$ of infinite order. Proof deferred.

Quasi-convexity, Morse property, and strongly-contracting:

We say a set $A \subset X$ has the *Morse property* if (λ, ϵ) -quasi-geodesics with endpoints in A stay $M(\lambda, \epsilon)$ -close to A . Another property is called *strongly-contracting*, which is that the closest-projection map $\pi_A : X \rightarrow A$ takes balls disjoint from A to sets of uniformly bounded diameter. By definition, Morse subsets are quasi-convex, but the converse is false in general (e.g. lines in $\mathbb{R}^2$). It's always true that strongly-contracting subsets are Morse, but the converse is false in general. In the δ -hyperbolic setting, all three properties are equivalent. Quasi-convexity implying Morse follows from stability of quasi-geodesics, and the implication that Morse implies strongly contracting is an exercise. Note that under quasi-isometries, only the Morse property is guaranteed to be always preserved (e.g. rays in $\mathbb{R}^2$ can be mapped to the logarithmic spiral under an quasi-isometry)

Non quasi-convex subgroups:

Recall that in $BS(1, 2) = \langle a, t : tat^{-1} = a^2 \rangle$, the subgroup $\langle a \rangle$ is distorted. Note that $BS(1, 2) = \langle a \rangle \rtimes \langle t \rangle$.

Question: Quasi-convex subgroups of G^{hyp} are hyperbolic, but what about non quasi-convex subgroups?

We know that a hyperbolic group cannot have distorted copies of cyclic groups, but it can have non quasi-convex subgroups.

Primary example:

- Let S be a genus $g \geq 2$ surface and let M_ϕ be the mapping torus of S with monodromy a pseudo-Anosov map $\phi : S \rightarrow S$. By the work of Thurston, $\pi_1(M_\phi) = \pi_1(S) \rtimes \mathbb{Z}$ is hyperbolic, but $\pi_1(S)$ is distorted.
- Note that in both $BS(1, 2)$ and $\pi_1(M_\phi)$, the non-quasi-convex subgroup comes from the kernel of some map to $\mathbb{Z}$. There are more general constructions of this flavor by I. Kapovich.

Question: What are properties of subgroups of hyperbolic groups G ? Are they always hyperbolic.

Question: If G has type F (i.e. has a finite $K(G, 1)$), and G does not contain any copies of $BS(m, n)$, is G hyperbolic?

Recall that type F is a generalization of finitely-generated and finite-presented. This assumption is necessary as Brady found examples of hyperbolic groups with non-hyperbolic finite-presented subgroups, hence these subgroups also do not contain any $BS(m, n)$. However, recently, Italiano-Martelli-Migliorini also found an example of a hyperbolic group that contains a non-hyperbolic subgroup of type F , which completely answers the latter question in the negative.

The following question remains open.

Question: If G is a one-relator group with no copies of $BS(m, n)$, is G hyperbolic?

7 Finite Subgroups of Hyperbolic Groups

Proposition 5. *There exist finitely many conjugacy classes of finite subgroups of a hyperbolic group G .*

Corollary 13. *There exist finitely many conjugacy classes of torsion elements in G^{hyp} .*

Proof. Here is a general strategy. Let $G \curvearrowright X$ be geometric, and let $GB = X$ for some bounded set B . Suppose that for every finite subgroup $H < G$, there exists a fixed point for H in X , i.e. some $x \in X$ such that $hx = x$ for all $h \in H$. Then, by coboundedness, there exists $g \in G$ such that $gx \in B$. Thus gHg^{-1} fixes gx , so $gHg^{-1} \subseteq \{g' \in G : g'(B) \cap B \neq \emptyset\}$. The latter is a finite set in G , by properness of action. Thus, the proposition holds.

The goal is now to produce fixed points for a finite group H , which is often guaranteed by the presence of non-positive curvature. Choose a base point $o \in X$ and define the set of centers-of-mass for the orbit Ho to be

$$C_H = \{x \in X : B(x, r_H) \supseteq Ho, \text{ where } r_H \text{ is the smallest possible radius}\}$$

C_H is clearly H -invariant, so if it's a singleton, then H has a fixed point. However, for G^{hyp} and $G \curvearrowright X$ geometrically, say $X = Cay(G, S)$. Then, fixed points do not always exist for H . However, one can show that $diam(C_H) \leq 4\delta$, which implies that for all $x \in C_H$, $d(x, hx) \leq 4\delta$. This means that finite subgroups of a hyperbolic group always have coarse fixed points, which is enough for the above argument to go through. $\square$

Question: Is every hyperbolic group G virtually torsion-free?

Question: Is every hyperbolic group G residually finite, i.e. $\forall g \neq 1 \in G, \exists \phi : G \rightarrow Q, Q \text{ finite group, such that } \phi(g) \neq 1$?

The above questions are equivalent to each other, and they are open problems.

8 Dehn's Algorithm

Assume $G = \langle S \rangle$. In the 1920's, Dehn posed three problems, the first two are stated below:

1. (The word problem) Is there an algorithm to decide if $w \in F(S)$ represents 1 in G ?
2. (The conjugacy problem) Is there an algorithm to decide if $u, v \in F(S)$ are conjugate in G ?

Dehn provides the following algorithm. Let $G = \langle S | R \rangle$; note that this exactly means $\pi_1(Cay(G, S)) = \langle\langle R \rangle\rangle$.

Take $w \in F(S)$.

Step 1: Freely reduce w by canceling terms that look like ss^{-1} . If by doing this we get the empty word, then stop. Otherwise, continue to step 2.

Step 2: Look for a subword $u \subset w$ which is a subword in a cyclic shift of some $r \in R$ such that $\|u\|_S > \frac{1}{2}\|r\|_S$. If there are no such u , then stop. Otherwise, replace u by v (possibly the empty word) such that $uv^{-1} = r$, so $\|v\| < \|u\|$. Stop if the resulting word is the empty word. Else repeat the entire process by going back to step 1.

The algorithm will always stop if $|S|, |R| < \infty$.

Definition 15. We say the algorithm solves the word problem for $G = \langle S | R \rangle$ if $w \stackrel{G}{=} 1 \iff$ the algorithm terminates at the empty word.

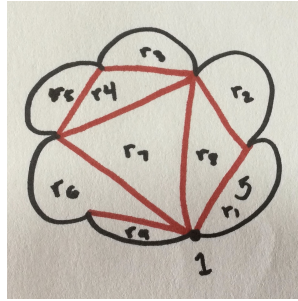
Theorem 4. The following are equivalent

1. G is a hyperbolic group
2. G has a finite presentation $\langle S | R \rangle$ for which the Dehn's algorithm solves the word problem (this is called a Dehn's presentation)
3. G is finitely presented and has a linear Dehn function
4. G is finitely presented and has a subquadratic Dehn function

Proof. (1 $\implies$ 2) Let $G = \langle S \rangle$ and say $\text{Cay}(G, S)$ is δ -hyperbolic. Let $R = \{u \in F(S) : \|u\|_S \leq 16\delta, u \stackrel{G}{=} 1\}$. We claim that $G = \langle S | R \rangle \iff$ Dehn's algorithm solves the word problem. For this, we need the following lemma.

Lemma 7. In a δ -hyperbolic space X , if $c : [a, b] \rightarrow X$ is a k -local geodesic for $k > 8\delta$ (i.e. $c|_{[s, s+k]}$ is a geodesic $\forall s \in [a, b - k]$), then $c(a) \neq c(b)$.

Suppose $w \stackrel{G}{=} 1$ is a loop. Then by the lemma, some segment u in w of length at most 8δ is not a geodesic segment, meaning its endpoints can be connected by a shorter segment v . The corresponding loop $r = uv^{-1}$ has length at most 16δ , so $r \in R$. We now apply the algorithm to replace u by v , yielding a shorter loop, which differs from the original loop (up to cyclic permutation) by some conjugate of r . Proceeding this way, we see that by the lemma, the algorithm will only terminate at the empty word, in which case we have also shown that $w \in \langle\langle R \rangle\rangle$.



We note that the assumption that G is δ -hyperbolic is necessary. This lemma does not hold in $\mathbb{Z}^2$. □

Corollary 14. All hyperbolic groups G are finitely presentable.

Corollary 15. A non-hyperbolic group must have at least a quadratic Dehn function.

Now, the conjugacy problem is harder than the word problem, but we can say something about it.

Definition 16. Let $f : \mathbb{N}^2 \rightarrow \mathbb{N}$. We say G satisfies that f -bounded conjugator property if for all $u, v \in G$, $u \stackrel{\text{conj.}}{\sim} v$, then there exists a $w \in G$, such that $wuw^{-1} = v$ and $\|w\| \leq f(\|u\|, \|v\|)$.

Proposition 6. If a finitely generated group $G = \langle S \rangle$ has solvable word problem and satisfies the f -bounded conjugator property for some f , then G has solvable conjugacy problem.

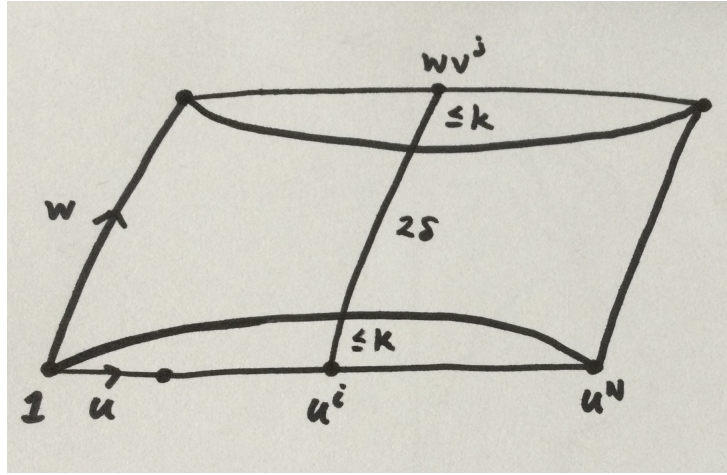
Proof. Let $u, v \in F(S)$. Consider $B = B(1, f(\|u\|, \|v\|))$ in $F(S)$. Since G satisfies the f -bounded conjugator property, $u \stackrel{\text{conj.}}{\sim} v \iff \exists$ a conjugator $w \in B$ for u and v . Note that $\#B < \infty$, so we can look for such a w by solving the word problem of the form:

$$wuw^{-1}v^{-1} \stackrel{G}{=} 1, \forall w \in B.$$

□

Theorem 5. Every hyperbolic group G satisfies the linearly-bounded conjugator property.

Proof. Assume $u \stackrel{\text{conj.}}{\sim} v$. If u, v are torsion, then we use the fact that there are only finitely many conjugacy classes of torsion elements and there exist coarse fixed points for u and v . If u and v are of infinite order, we get the following picture.



If w is the conjugator, then

$$d(1, u^{-i}wv^j) = d(u^i, wv^j) \leq k_u + k_v + 2\delta,$$

where k_u and k_v are the quasi-convexity constants for u and v . It's not hard to see that where k_u depends only on $\|u\|$ linearly. □

Other finiteness properties are obtained via the Rips Complex:

- Torsion free hyperbolic groups G have type F , finite $K(G, 1)$
- More generally, hyperbolic groups G always have type F_∞ , $K(G, 1)^k$ is finite for all k .

9 Gromov Boundary

Let X be δ -hyperbolic and $o \in X$.

Definition 17. Define the Gromov boundary to be

$$\partial_o X = \{\gamma : [0, \infty) \rightarrow X \mid \text{geodesics } \gamma(0) = o\} / \sim$$

where $\gamma \sim \gamma'$ if $\sup_t d(\gamma(t), \gamma'(t)) < \infty$.

We get the following inclusions:

$$\partial_0(X) \hookrightarrow \partial X = \{\gamma \text{ geodesic rays}\} / \sim \hookrightarrow \partial_q X = \{\gamma \text{ quasi-geodesic rays}\} / \sim$$

Proposition 7. The above maps are bijective.

The topology on this space is the following: for $\xi \in \partial_0 X$ and $r \in \mathbb{R}$,

$$B(\xi, r) = \{\text{rays } \alpha' : \alpha(0) = \alpha'(0), d(\alpha(t), \alpha(t')) < k, t \in [0, r]\}$$

The Gromov boundary has the following properties. Let X be a δ -hyperbolic proper metric space.

1. ∂X is compact, metrizable, and $\bar{X} = X \cup \partial X$ is a compactification of X .
2. $G \curvearrowright X$ by isometries induces $G \curvearrowright \partial X$ by homeomorphism
3. If X, Y are δ -hyperbolic, then a quasi-isometric embedding $f : X \rightarrow Y$ induced a topological embedding $\partial f : \partial X \rightarrow \partial Y$. $X \overset{\text{Q.I.}}{\cong} Y \implies \partial X \cong \partial Y$.

Definition 18. $\partial G^{\text{hyp}} \partial X$ for $G \curvearrowright X$ geometrically

Examples:

- X bounded, $\partial X = \emptyset$
- $\partial \mathbb{R} = \{\pm\infty\}$
- $\partial \mathbb{H}^n = S^{n-1} \implies \mathbb{H}^n \overset{\text{Q.I.}}{\not\cong} \mathbb{H}^m, n \neq m$
- $\partial \pi_1(S_g) \cong S^1, g \geq 2$
- $\partial \mathbb{F}_n = \text{cantor set}$

We now want to classify the isometries of a hyperbolic space X . Take $g \in \text{Isom}(X)$. Define the stable translation of g to be

$$\tau_g = \lim_{n \rightarrow \infty} \frac{d(x, g^n x)}{n}$$

which is independent of choice of $x \in X$.

Theorem 6. Every $g \in \text{Isom}(X)$ is either

1. elliptic: has a bounded orbit $\iff g$ has a coarse fixed point in $X \implies \tau_g = 0$
2. parabolic: has an unbounded orbit and $\tau_g = 0 \iff$ has a unique fixed point in ∂X

3. *loxodromic*: has an unbounded orbit and $\tau_g > 0 \iff g$ has a quasi-geodesic axis $\iff g$ has 2 fixed points in ∂X

Corollary 16. *If G is hyperbolic, then every infinite order $g \in G$ is loxodromic.*

Proof. $\{g^n\}_{n \in \mathbb{Z}}$ is quasi-geodesic. □

Theorem 7. *If G is hyperbolic, then either*

1. G is finite $\iff \partial G = \emptyset$,
2. G is virtually $\mathbb{Z}$ $\iff \partial G = \{\pm\infty\}$, or
3. $G \geq \mathbb{F}_2 \iff |\partial G| = \infty$.

Proposition 8. *If $|G^{hyp}| = \infty$, then G contains some infinite order element g .*

Proof. Fix $G = \langle S \rangle$, and let η, ξ be two distinct points in $\partial G = \partial \text{Cay}(X, S)$. (It's an exercise that ∂G can't be a singleton when G is infinite.) By Arzela-Ascoli, we can find a biinfinite geodesic $\gamma : \mathbb{R} \rightarrow \text{Cay}(G, S)$ with endpoints ξ and η . Let $K = d(1, \text{im}(\gamma))$. Let h and g be vertices along $\gamma(t)$ such that $d(1, h) \gg K$ and $d(1, g) \gg K$, where h and g are "close" to ξ and η respectively. If h or g is infinite order, then we are done. Else, $h^{-1}g$ is infinite order. □

Proof of Theorem 7. If $|\partial G| \geq 3$, then we can find a pair of independent loxodromics g, h , i.e. $\partial g \cap \partial h = \emptyset$. Then, apply the ping-pong lemma to get $\langle g^N, h^N \rangle \cong \mathbb{F}_2$, for some N . □

Theorem 8. *(Strong Tits Alternative) Every $H < G^{hyp}$ is either virtually cyclic or $H > \mathbb{F}_2$.*

We recall that $\partial\pi_1(S_g) \cong S^1$.

Theorem 9. *If G is hyperbolic with $\partial G \cong S^1$, then G is a Fuchsian group (i.e. virtually $\pi_1(S_g)$ for some $g \geq 2$).*

This theorem is due independently by Tukia, Casson-Jungreis, and Galai. Tool: the group action $G \curvearrowright \partial G$ is a convergence group action, that is G acts properly discontinuously on distinct triples of points in ∂G .

Cannon Conjecture: Suppose $\partial G \cong S^2$, then G is a Kleinian group.

Gromov Conjecture: If G is one-ended, then $G > \pi_1(S_g)$ for some $g \geq 2$.

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